SL Paper 2

Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} \lambda & 3 & 2 \\ 2 & 4 & \lambda \\ 3 & 7 & 3 \end{bmatrix}.$$

Suppose now that $\lambda = 1$ so consider the matrix

$$\boldsymbol{B} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ 3 & 7 & 3 \end{bmatrix}.$$

a.i. Find an expression for det(***A***) in terms of λ, simplifying your answer. [3]

a.ii. Hence show that ***A*** is singular when $\lambda = 1$ and find the other value of λ for which ***A*** is singular. [2]

b.i. Explain how it can be seen immediately that ***B*** is singular without calculating its determinant. [1]

b.ii. Determine the null space of ***B***. [4]

b.iii. Explain briefly how your results verify the rank-nullity theorem. [[N/A

c. Prove, using mathematical induction, that [7]

$$\boldsymbol{B}^n = 8^{n-2}\boldsymbol{B}^2 \text{ for } n \in \mathbb{Z}^+,\ n \geqslant 3.$$

The hyperbola with equation $x^2 - 4xy - 2y^2 = 3$ is rotated through an acute anticlockwise angle α about the origin.

a. The point $(x,\ y)$ is rotated through an anticlockwise angle α about the origin to become the point $(X,\ Y)$. Assume that the rotation can be represented by [3]

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Show, by considering the images of the points $(1,\ 0)$ and $(0,\ 1)$ under this rotation that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

b.i. By expressing $(x,\ y)$ in terms of $(X,\ Y)$, determine the equation of the rotated hyperbola in terms of X and Y. [3]

b.ii. Verify that the coefficient of XY in the equation is zero when $\tan\alpha = \frac{1}{2}$. [3]

b.iii. Determine the equation of the rotated hyperbola in this case, giving your answer in the form $\frac{X^2}{A^2} - \frac{Y^2}{B^2} = 1$. [3]

b.iv Hence find the coordinates of the foci of the hyperbola prior to rotation. [5]

S is defined as the set of all 2×2 non-singular matrices. A and B are two elements of the set S.

a. (i) Show that $(A^T)^{-1} = (A^{-1})^T$. [8]

(ii) Show that $(AB)^T = B^T A^T$.

b. A relation R is defined on S such that A is related to B if and only if there exists an element X of S such that $XAX^T = B$. Show that R is an equivalence relation. [8]

The matrix $\boldsymbol{A}$ is given by $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 8 & 11 & 8 \\ 1 & 3 & 4 & \lambda \\ \lambda & 5 & 7 & 6 \end{pmatrix}$.

(a) Given that $\lambda = 2$, $\boldsymbol{B} = \begin{pmatrix} 2 \\ 4 \\ \mu \\ 3 \end{pmatrix}$ and $\boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$,

(i) find the value of μ for which the equations defined by $\boldsymbol{AX} = \boldsymbol{B}$ are consistent and solve the equations in this case;

(ii) define the rank of a matrix and state the rank of $\boldsymbol{A}$.

(b) Given that $\lambda = 1$,

(i) show that the four column vectors in $\boldsymbol{A}$ form a basis for the space of four-dimensional column vectors;

(ii) express the vector $\begin{pmatrix} 6 \\ 28 \\ 12 \\ 15 \end{pmatrix}$ as a linear combination of these basis vectors.

The set of all permutations of the list of the integers $1,\ 2,\ 3\ \ldots\ n$ is a group, S_n, under the operation of composition of permutations.

Each element of S_4 can be represented by a 4×4 matrix. For example, the cycle $(1\ 2\ 3\ 4)$ is represented by the matrix

$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ acting on the column vector $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

a. (i) Show that the order of S_n is $n!$; [9]

(ii) List the 6 elements of S_3 in cycle form;

(iii) Show that S_3 is not Abelian;

(iv) Deduce that S_n is not Abelian for $n \geqslant 3$.

b. (i) Write down the matrices $\boldsymbol{M}_1, \boldsymbol{M}_2$ representing the permutations $(1\ 2),\ (2\ 3)$, respectively; [7]

(ii) Find $\boldsymbol{M}_1\boldsymbol{M}_2$ and state the permutation represented by this matrix;

(iii) Find $\det(\boldsymbol{M}_1), \det(\boldsymbol{M}_2)$ and deduce the value of $\det(\boldsymbol{M}_1\boldsymbol{M}_2)$.

c. (i) Use mathematical induction to prove that [8]

$(1\ n)(1\ n\ -1)(1\ n-2)\ldots(1\ 2) = (1\ 2\ 3\ldots n)\ n \in \mathbb{Z}^+,\ n > 1.$

(ii) Deduce that every permutation can be written as a product of cycles of length 2.

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a. Given that the elements of a 2×2 symmetric matrix are real, show that [11]

(i) the eigenvalues are real;

(ii) the eigenvectors are orthogonal if the eigenvalues are distinct.

b. The matrix $\boldsymbol{A}$ is given by [7]

$$\boldsymbol{A} = \begin{pmatrix} 11 & \sqrt{3} \\ \sqrt{3} & 9 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of $\boldsymbol{A}$.

c. The ellipse E has equation $\boldsymbol{X}^T\boldsymbol{AX} = 24$ where $\boldsymbol{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\boldsymbol{A}$ is as defined in part (b). [7]

(i) Show that E can be rotated about the origin onto the ellipse E' having equation $2x^2 + 3y^2 = 6$.

(ii) Find the acute angle through which E has to be rotated to coincide with E' .

The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is defined by $\boldsymbol{X} \mapsto \boldsymbol{AX}$, where $\boldsymbol{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a , b , c , d are all non-zero.

Consider the group $\{S, +_m\}$ where $S = \{0, 1, 2\ldots m-1\}$, $m \in \mathbb{N}$, $m \geq 3$ and $+_m$ denotes addition modulo m .

A.a Show that f is a bijection if $\boldsymbol{A}$ is non-singular. [7]

A.b Suppose now that $\boldsymbol{A}$ is singular. [5]

(i) Write down the relationship between a , b , c , d .

(ii) Deduce that the second row of $\boldsymbol{A}$ is a multiple of the first row of $\boldsymbol{A}$.

(iii) Hence show that f is not a bijection.

B.a Show that $\{S, +_m\}$ is cyclic for all $\boldsymbol{m}$. [3]

B.b Given that m is prime, [7]

(i) explain why all elements except the identity are generators of $\{S, +_m\}$;

(ii) find the inverse of x , where $\boldsymbol{x}$ is any element of $\{S, +_m\}$ apart from the identity;

(iii) determine the number of sets of two distinct elements where each element is the inverse of the other.

B.c. Suppose now that $m = ab$ where a , b are unequal prime numbers. Show that $\{S, +_m\}$ has two proper subgroups and identify them. [3]

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a. By considering the points $(1,\ 0)$ and $(0,\ 1)$ determine the 2×2 matrix which represents [5]

(i) an anticlockwise rotation of θ about the origin;

(ii) a reflection in the line $y = (\tan\theta)x$.

b. Determine the matrix A which represents a rotation from the direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to the direction $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. [2]

c. A triangle whose vertices have coordinates $(0,\ 0)$, $(3,\ 1)$ and $(1,\ 5)$ undergoes a transformation represented by the matrix $A^{-1}XA$, where X is the matrix representing a reflection in the x-axis. Find the coordinates of the vertices of the transformed triangle. [6]

d. The matrix $B = A^{-1}XA$ represents a reflection in the line $y = mx$. Find the value of m. [6]
